Dealing with Loops

19CSE205 : PROGRAM REASONING

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Contents

A simple looping program

Computing sum of first n integers.

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- Frama-c fails to prove.
- But we don't know why?
- Could it be because of loop?
- Let's first confirm. **Test**!

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- Loop is re-written for fixed n.
- In this case n = 3.
- The underlying logic is same.
- Frama-c is able to prove the correctness now.
- Note the postcondition remains the same.

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Deduction seems to breakdown in the presence of loops.
Two problems are evident.

1. Loops are unbounded

How can one be sure the contract will be satsified for any n?

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Does P' ⇒ Q'?

$x < 5$

x<2 ⇒ x< 5? ✓

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P': $x<2$ Q': $x<5$

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$x < 4$

$x = x + 1;$

$x < 5$

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$x<2 \Rightarrow x< 4?$ ✓

$\uparrow$

$x<2 \Rightarrow x< 5?$ ✓

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$x < 3$	$x<2 \Rightarrow x< 3$? ✓
$x = x + 1;$	$\uparrow$
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$x < 2$	$x<2 \Rightarrow x< 2$? ✓
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2. Loop may run forever

What is the guarantee that the loop will eventually terminate?

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while ( i < n ) {
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    ......
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```

- i never gets incremented.

2. Loop may run forever

What is the guarantee that the loop will eventually terminate?

```
while ( i < n ) {
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- i never gets incremented.

```
while ( i != n ) {
    ......
    i = i + 1
    n = n + 1
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```

- n increases along with i.

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while ( i != n ) {
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i = 1
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Bottomline:

- The programmers can write their code in any manner.
- They can state the input and output conditions in any way.
- The proof system must not make any assumptions about the code.
- Proof construction must be based on generic principles.

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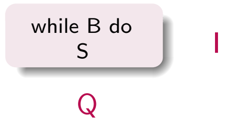
- This magic property is called loop invariant.

  i.e. $I = wp(S,I)$ where I is the loop invariant.



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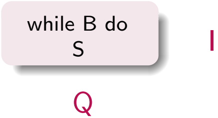
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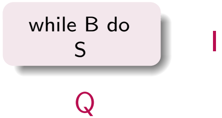

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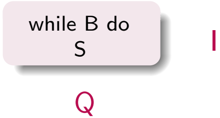

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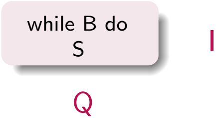

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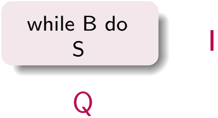

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\begin{aligned}
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How do we come up with this loop invariant?

Any thumb rules?

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Loop invariant must capture the progress made as iterations proceed.

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Loop invariant must capture the progress made as iterations proceed.

Loop invariant must capture the span of entry and exit condition range.

Let's apply this to the example

File: sigma-loop.c

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*/
int sigma(int n) {
    int s = 0;
    int i = 1;
    /*@



    */
    while (i <= n) {
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1. Capturing progress

Evaluation of while entry condition	i	s
Before iteration 1	1	0
Before iteration 2	2	1
Before iteration 3	3	3
Before iteration 4	4	6
. . .	..	..
. . .	..	..
Before iteration n	n	(n-1)*n/2
After iteration n	n+1	n*(n+1)/2

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Progress made is captured by (i−1)*i/2.

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    /*@
      loop invariant s == (i−1)*i/2;


    */
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Before iteration 4	4	6
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Before iteration n	n	(n-1)*n/2
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Loop invariant must capture the span of entry and exit condition range.

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Variations to try

Apply these variations to sigma program to improve your understanding.

1. Remove loop invariant s == (i−1)*i/2;
2. Remove loop invariant 1 <= i <= n+1;
3. Replace (i−1)*i/2 with i*(i+1)/2 in first loop invariant.
4. Replace loop invariant 1 <= i <= n+1; with loop invariant i <= n+1;
5. Replace loop invariant 1 <= i <= n+1; with loop invariant 1 <= i <= n;
6. Do as in bullet 4. In addition, replace while (i <= n) with while (i < n).
7. Remove the statement i = i + 1;
8. Replace loop invariant 1 <= i <= n+1; with loop invariant 1 <= i <= n+2;. Now, modify your program such that criteria is met but program is wrong.
9. Re-write the while loop to iterate in reverse way. i.e. n + (n−1) + . . . + 1. What changes would you have to make to prove all goals?

Follow these instructions when you try the variations.

- Implement one variation at a time and reason out the frama-c output.
- Run frama-c-gui -wp ⟨program⟩ to see which goal cannot be proved.
- Don't make silly errors and waste time resolving them. Focus on checking logic.